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Smooth Limits of Piecewise-Linear Approximations

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We consider particular types of discrete approximations to tensor fields on manifolds suggested by triangulations. The approximations are objects of finite geometrical extent, parameterized by a finite set of numbers, so they are suitable for numerical computations. We study the limiting behaviour of sequences of approximations and construct the theory so that the limits are tensor fields on the manifold. We propose a Cauchy criterion for our approximations, which guarantees convergence to a limit. The specific examples include geodesic approximation to Riemannian and pseudo-Riemannian manifolds.

1. INTRODUCTION

We consider particular types of discrete approximations to tensor fields on manifolds suggested by triangulations. The approximations are objects of finite geometrical extent and are parameterized by a finite set of numbers, so they are suitable for numerical computations. We study the limiting behaviour of sequences of approximations and construct the theory so that the limits are tensor fields on the manifold.

In numerical applications, one does not know the limiting geometry. We propose a Cauchy criterion for our approximations, which guarantees convergence to a limit.

A key feature of our investigation is the "tensorial character" of our approximations. By this, we mean that our approximations have been designed so that they behave well under changes of coordinates on the manifold. We verify this in detail. Thus we believe that in an application,

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one could change the coordinates of the manifold during a numerical computation.

The original motivation for our work was to study approximations for general relativity. One of the issues in this subject is the proper handling of coordinate changes, particularly for numerical computations of gravitational collapse. We refer the reader to [4] for an overview of the current status of approximation methods in general relativity.

One of the long-term goals, not realised in this paper, is to remove the smooth manifold structure from the approximation procedure, but to be able to obtain it as part of the calculation of the limit.

Our work started with the particular case of Riemannian metric tensors. The metric triangulations of Regge calculus [1, 10] generate the discrete approximations we have in mind. The work of Cheeger, Müller, and Schrader (CMS) [3] shows that given a Riemannian metric, one can generate some very particular sequences of metric triangulations which contain all the information in the scalar curvatures of the original Riemannian metric. The work of CMS defines the limit of scalar curvatures only, these being all functions, not tensors.

We show that the sequences of CMS satisfy our Cauchy criterion; hence the entire metric tensor is obtained in the limit. Thus the sequences of approximations actually contain all the geometric information.

We mention a second long-term goal, also not realised in this paper, which is to perform a limiting process for curvature tensors which is entirely local, or pointwise.

In Section 2, we investigate Riemannian metric tensors without introducing any theoretical superstructure. We hope that this provides some motivation for the abstractions to be encountered in Section 3. The main result (Theorem 2.2) shows that approximations to a Riemannian metric, such as those obtained from [3], converge to that Riemannian metric. The keys to this theorem are Lemmas 2.4 and 2.7, which also play important roles in Section 3.

The generalization we develop in Section 3 moves away from purely metric data to any type of tensor field. In doing so, we are able to handle metric tensors of indefinite signature, for example. After some preparations, we introduce our Cauchy criterion (Definition 3.9) and show that it gives rise to well-defined tensor fields on the entire manifold (Corollary 3.11). We continue with our notion of uniformly Cauchy (Definition 3.12) and verify that it behaves as one would expect (Theorem 3.14). Finally, we show that the particular sequences of metric triangulations used in [3] are uniformly Cauchy in our sense (Theorem 3.15).

The convergence remains essentially pointwise, so does not approximate tensors of a weaker, distributional type [8]. Subsequent investigations will consider distributional metric tensors.

Some examples are presented in Section 4. These are mainly in the nature of counterexamples, which show that removing any one of our hypotheses results in ill-defined limiting behavior. Example 4.7 treats the case of indefinite signature, important for our main motivating application to relativity.

Finally, we mention that this work can be taken to support a certain thesis about discreteness in fundamental physics: that the continuum metric tensor of spacetime can be regarded as a limit of some discrete data of a physical kind, such as distance measurements, in the spirit of Ehlers *et al.* [5] and Marzke and Wheeler [6].

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2. Riemannian Metrics

First, we explain some conventions which are used throughout the paper. For a tensor t on a set S, we define $|t|_\infty$ to be the supremum over S of the absolute values of the components of t in the current local coordinates. Usually, the set S and the current local coordinates will be clear from context. One particular case of interest will be when S is a single point.

We first consider a concrete case to help motivate the general constructions in Section 3. Let (M, g) be a Riemannian manifold of dimension D and denote the distance function (Fréchet metric) by d.

Definition 2.0. A piecewise-flat (PF) metric on a simplicial complex is a Riemannian metric on each simplex having constant components in the standard coordinates for the simplex. These are such that if σ_1 is a face of σ_2, then the metric on σ_1 is that induced from σ_2 by inclusion. For later use, a piecewise-flat pseudo-Riemannian metric of a particular signature is defined in the same way.

In the case of a PF metric, we denote the volume of a simplex by $|\sigma|$. The dimension of a simplex is often denoted by a superscript, e.g., σ^D. The mesh of a complex K is, as usual, $\eta = \sup\{|\sigma^1|, \sigma^1 \in K\}$.

DEFINITION 2.1. A piecewise-flat (PF) approximation to (M, g) with error $f \in \mathbf{R}$ is a triangulation $\varphi: K \to M$ together with a PF metric on K. These are such that if $|\sigma^1|$ is an edge length in the PF manifold K and $\sigma^1 = (v_1, v_2)$, then $|\,|\sigma^1| - d(\varphi(v_1), \varphi(v_2))| \leqslant f$.

Remark. The definition of approximation used here is slightly more general than that of [3] in that we have allowed an error of f. This increase in generality allows the consideration of approximations which satisfy some local equations.

Let σ be a simplex of K. Recall [3] that the *fatness* of σ relative to the mesh of K is

$$\theta(\sigma) = \inf_{\text{faces } \tau} \frac{|\tau|}{\eta^{\dim \tau}},$$

and the fatness of K is

$$\theta = \inf_{\sigma \in K} \theta(\sigma).$$

Our first main result is

THEOREM 2.2. *Let $f: \mathbf{R}^+ \to \mathbf{R}$ obey $f(\eta) = o(\eta)$. If $\{\varphi_{(i)}: K_{(i)} \to M\}$ is a sequence of PF approximations to (M, g) with errors $f(\eta_{(i)})$ such that*

(i) *the fatness $\theta_{(i)}$ is bounded away from zero,*

(ii) *$\eta_{(i)} \to 0$ as $i \to +\infty$, and*

(iii) *the set of the images of all the vertices of all the triangulations is dense in M,*

then the metric tensor g on M is uniquely determined by the edge-length data. In fact, for each $p \in M$, the PF approximations determine sequences of metrics on T_pM which converge to the value of g at p.

As our first step in the proof of Theorem 2.2, we consider the geodesic equation

$$\frac{d\xi^i}{dt} + \Gamma^i_{jk} \xi^j \xi^k = 0$$

for $0 \leqslant t \leqslant 1$ and seek a bound on $\xi(1)$ in terms of $\xi(0)$, the distance d between the endpoints of the geodesic, and other constants. Let $\zeta = \xi - \xi(0)$ and compute

$$\frac{d\,|\zeta(t)|_\infty}{dt} \leqslant \lambda \frac{ds}{dt} |\zeta + \xi(0)|_\infty,$$

where λ is a constant which depends only on the bounds on the connection coefficients and the inverse metric in the $|\cdot|_\infty$ norm, and s is the geodesic distance. Integrating, we obtain

LEMMA 2.3. *With the preceding notation,*

$$\frac{|\xi(1)-\xi(0)|_\infty}{|\xi(0)|_\infty} \leqslant e^{\lambda d} - 1.$$

Clearly, the right-hand side can be made as small as one likes by choosing d to be sufficiently small. This accords with one's intuition that sufficiently small regions deviate little from flatness.

LEMMA 2.4. *Let $V \subset M$ and $U \subset \mathbf{R}^D$ be open sets and $\chi: V \to U$ a C^2 diffeomorphism. Let h_V be a C^∞ metric on V. For any point $q \in U$ and any $\varepsilon > 0$ there is a smaller neighbourhood $U' = \chi(V')$ of q such that (V', h_V) is convex and*

$$|d_V(a,b)^2 - (\chi_* h_V)(q)(\chi(a)-\chi(b))| \leqslant \varepsilon\, |\chi(a)-\chi(b)|_\infty^2,$$

for all points a and b in V'.

The notation $\chi_*(h_V)$ denotes the push-forward of the metric, effectively a coordinate description of the metric, while $(\chi_* h_V)(q)$ denotes its value at q, which can be regarded as a flat Euclidean metric on the entire $\mathbf{R}^D$. A metric with a single argument is to be regarded as a quadratic form.

Proof. We use a coordinate notation for the quantities defined on $U \subset \mathbf{R}^D$. Thus we write g_{ij} for the components of $\chi_* h_V$, and x^i for the components of $\chi(a)-\chi(b)$. Similarly, Γ^i_{jk} denotes the connection components of g_{ij}.

Note first of all that it is possible to find a smaller convex $V'' \subset V$ such that g_{ij}, its inverse, and Γ^i_{jk} are bounded on $U'' = \chi(V'')$. Consider a, b in this set.

Letting ξ be the tangent vector field along the geodesic of $\chi_* h_V$ from $\chi(a)$ to $\chi(b)$,

$$\begin{aligned} d(a,b) &= \int_0^1 \sqrt{g_{ij}\xi^i\xi^j}\, dt \\ &= \sqrt{g_{ij}\xi^i\xi^j}\,(c) \end{aligned}$$

for some c on the geodesic between $\chi(a)$ and $\chi(b)$ by the mean value theorem. Thus

$$\begin{aligned} d(a,b)^2 &= g_{ij}(c)\, x^i x^j + e \\ &= g_{ij}(q)\, x^i x^j + e + e', \end{aligned}$$

where e and e' are the respective error terms.

We use the result of Lemma 2.3, and for brevity write μ for the small parameter $e^{\lambda d(a,b)} - 1$. Using the formula $x^1 = \int_0^1 \xi^i \, dt$, one can show that

$$|x - \xi|_\infty \leqslant \mu \, |\xi|_\infty, \tag{*}$$

and hence that $|\xi|_\infty \leqslant 1/(1-\mu) \, |x|_\infty$. Since

$$e = g_{ij}(c)(\xi^i(c) - x^i)(\xi^j(c) + x^j),$$

it follows that

$$|e| \leqslant D^2 \, |g(c)|_\infty \, |x|_\infty^2 \, \frac{\mu(2-\mu)}{(1-\mu)^2}.$$

The other error term is

$$|e'| \leqslant D^2 \, |g(q) - g(c)|_\infty \, |x|_\infty^2.$$

To complete the proof, it remains to define $V' \subset V''$ so that $(V', h_{V'})$ is convex and has sufficiently small diameter that

$$D^2 \, |g|_\infty \, \frac{\mu(2-\mu)}{(1-\mu)^2} + D^2 \, |g(q) - g(c)|_\infty \leqslant \varepsilon$$

for all $c \in U' = \chi(V')$, with $\mu = e^{\lambda \, \mathrm{diam}(V')} - 1$. This clearly can be done since g is continuous. ∎

The key estimate (*) is actually valid for arbitrary linear connections; a more sophisticated proof which gives this result is sketched near the end of Example 4.7.

Now consider $\mathbf{R}^D$ with a constant metric tensor γ. Let σ be any D-simplex in $\mathbf{R}^D$ and let Σ be a standard D-simplex with edges of γ-length 1. We write $|\sigma^1|$ and $|\sigma|$ for the γ-length and γ-volume, respectively, for example. Assume that for all edges σ^1 of σ, $|\sigma^1| \leqslant \lambda$, and that the fatness (relative to γ) $\theta(\sigma) \geqslant B > 0$. Let A be an affine map establishing a homeomorphism $\Sigma \to \sigma$ and denote its linear part by L.

PROPOSITION 2.5. *There are universal bounds, $\lambda b_1 \leqslant L \leqslant \lambda b_2$, with b_1, b_2 depending at most on B, where the inequalities are with respect to the operator norm defined by γ.*

Proof. By polar decomposition, write $L = SQ$, where S is positive-definite symmetric and Q is orthogonal. Pick a basis $\{w_i\}$ with $w_i = Q(v_i)$, where the v_i are the displacements along the edges of Σ with one vertex in common. Denoting the dual basis by $\{\bar{w}_i\}$, we have $\mathrm{tr}\, S = \sum_1 \bar{w}_i(Sw_i) \leqslant k\lambda$, where k is a numerical factor depending only on D according to

LEMMA 2.6. *Let v_i be the vectors along the edges of a simplex σ with a common vertex x, let $\bar{v}_i$ be the dual basis, and let τ be the $(D-1)$-face spanned by $\{v_2, v_3, ..., v_D\}$. Then for the γ-norm we have*

$$|\bar{v}_i| = \frac{|\tau|}{D\,|\sigma|}.$$

The proof is an easy calculation using the γ-perpendicular, which we leave to the reader.

Continuing the proof of Proposition 2.5, bounding tr S gives a bound on the spectrum of S, hence an upper bound for L. Thus b_2 does not depend on B.

For the lower bound, consider the inverse map $L^{-1} = Q^{-1}S^{-1}$ and apply the same argument to obtain a bound on tr S^{-1}. This time, let $\{w_i\}$ be a basis of displacements along the edges of σ having one vertex in common. Then $|S^{-1}w_i| = 1$ and $|\bar{w}_i| = |\tau_i|/D|\sigma|$. Now $|\tau| \leqslant k'\lambda^{D-1}$, where

$$k' = \frac{\sqrt{D+1}}{2^{D/2}D!}$$

is the volume of the largest simplex with edge-lengths $\leqslant 1$. Also, $|\sigma| \geqslant B\lambda^D$ whence $|\bar{w}_i| \leqslant k'/BD\lambda$. Thus, tr $S^{-1} \leqslant k'/B\lambda$ yielding a lower bound on the spectrum of L with $b_1 = B/k'$. ∎

We now have almost everything we need for the first main result.

Proof of Theorem 2.2. Let $p \in M$, fix a coordinate chart (V, χ) as in Lemma 2.4, and set $q = \chi(p)$. For each member of the sequence of PF manifolds, choose a D-simplex $\sigma_{(i)}$ so that the vertices converge to p as $i \to +\infty$.

Put the constant metric tensor $\gamma = \chi_* g(q)$ on the coordinate space $\mathbf{R}^D$. We show that the PF approximations allow one to define a sequence of constant metric tensors on $\mathbf{R}^D$ and that these converge to γ. This is enough to prove the theorem, as the coordinate mapping χ allows us to regard the convergence to γ as occurring in $T_p^*M \otimes T_p^*M$. The sequence is not uniquely determined.

For all $\varepsilon > 0$ there is a neighbourhood V' of p as in Lemma 2.4, and the vertices of all simplices are eventually in V'. According to Lemma 2.4, the distances between vertices measured with γ are well approximated by distance measurements with the curved metric g, and from Definition 2.1 these in turn are well approximated by the PF edge-lengths. Precisely, for $a, b \in V'$

$$|d_V(a, b)^2 - \gamma(\chi(a) - \chi(b))| \leqslant \varepsilon\,|\chi(a) - \chi(b)|_\infty^2.$$

For a suitable edge σ^1, $|\sigma^1| - d_V(a, b) = o(\eta)$ whence $|\,|\sigma^1|^2 - d_V(a, b)^2| = o(\eta^2)$. Combining these, we obtain

$$|\,|\sigma^1|^2 - \gamma(\chi(a) - \chi(b))| \leqslant \varepsilon(|\chi(a) - \chi(b)|_\infty^2 + \eta^2),$$

where a and b are images under $\varphi_{(i)}$ of vertices of $\sigma_{(i)}$, and i is sufficiently large. In this way, we have eliminated the need for further consideration of the curved manifold. We now compare flat metrics directly. To transfer fatness bounds and obtain convergence, we use the following.

LEMMA 2.7. *If $g_{(i)}$ is a sequence of constant positive-definite metrics, $A_{(i)}: \mathbf{R}^D \to \mathbf{R}^D$ are affine mappings, γ is a constant positive definite metric on $\mathbf{R}^D$, $\sigma_{(i)} \subset \mathbf{R}^D$ are simplices, for every $\varepsilon > 0$ and for all sufficiently large i*

$$|g_{(i)}(a - b) - \gamma(A_{(i)}a - A_{(i)}b)| < \varepsilon(g_{(i)}(a - b) + \gamma(A_{(i)}a - A_{(i)}b)),$$

where a and b are vertices, and we have the fatness bound $\theta_{g(i)}(\sigma_{(i)}) \geqslant B > 0$, then for sufficiently large i, $A_{(i)}$ is invertible and $\theta_\gamma(A_{(i)}(\sigma_{(i)})) \geqslant B' > 0$, and moreover $A_{(i)*}\, g_{(i)} \to \gamma$.

Proof. Without loss of generality, we may assume that $g_{(i)}$ is a constant sequence of positive-definite bilinear forms. This is because $GL(D)$ transformations may be applied to $g_{(i)}$, $\sigma_{(i)}$, and $A_{(i)}$. Therefore we set $g_{(i)} = g'$.

Our assumed inequality amounts to

$$|(g' - A^*_{(i)}\gamma)(e_{(i)})| < \varepsilon(g' + A^*_{(i)}\gamma)(e_{(i)})$$

for edge vectors $e_{(i)}$. For a fixed simplex, the $\{e \otimes e\}$, with e running over the edges, form a basis for the symmetric tensors. As Proposition 2.5 makes clear, these bases differ by linear maps which are bounded with respect to i, apart from an overall scaling which cancels on both sides of our estimate. Therefore, the estimate implies $\|g' - A^*_{(i)}\gamma\| < \varepsilon' \, \|g'\|$ for a suitable norm $\|\cdot\|$ on the space of symmetric matrices, where ε' can be made arbitrarily small by choosing ε sufficiently small. In a small enough neighbourhood of g', all matrices are positive-definite, so the $A_{(i)}$ are invertible. The convergence $A_{(i)*}\, g_{(i)} \to \gamma$ follows immediately and the fatness bounds are a consequence of continuity. ▮

Recalling that all norms are equivalent on finite-dimensional vector spaces, this includes the proof of Theorem 2.2 as well. ▮

3. General Tensors

We now generalize the apparatus considerably to handle tensors of any type, in particular, indefinite metric tensors. The resulting notions are somewhat abstract. The basic entities are convergent sequences of locally defined objects (Definitions 3.1, 3.4, 3.7, and 3.9). Note that our notion of convergence includes an eventual fatness bound. Through Lemma 3.5, this essentially is the source of the good behavior to be found in Proposition 3.8, Corollary 3.11, and Theorems 3.14 and 3.15.

In Section 2, the Riemannian metric played two roles: it was tensor data and it provided fatness bounds. Here we have separated these, allowing arbitrary tensor data, but keeping the weaker aspect of providing fatness bounds intact. The formulation of the latter is subtly changed, moving the location of the fatness bounds from the simplices to charts of local coordinates.

We begin with the locally defined objects.

Definition 3.1. A *local PL tensor* on M is a triple (σ, t, φ) in which $\sigma = (v_0, ..., v_D)$ is a geometric D-simplex in $\mathbf{R}^D$, t is a constant tensor on σ, and φ: $\{v_0, ..., v_D\} \to M$ has distinct images. When t is a metric tensor, we refer to the triple as a *local PL metric tensor*.

Definition 3.2 A local PL Riemannian metric $(\sigma, g_\sigma, \varphi)$ is a *local PF approximation* to the Riemannian manifold (M, g) with error f if and only if $|\,|(v_i, v_j)| - d(\varphi(v_i), \varphi(v_j))| \leqslant f$, where $|\tau|$ denotes the volume of the subsimplex $\tau \subseteq \sigma$ with respect to g_σ.

Definition 3.3. Let $\psi: U \to \mathbf{R}^D$ be a coordinate chart at $p \in M$. The *induced tensor* $\hat{t}$ on $\mathbf{R}^D$ is the push-forward of t via the map $(\psi \circ \varphi)^\wedge$ obtained by composing ψ with φ and extending the composition linearly to the whole simplex σ. We denote the image of σ under this map by $\hat{\sigma}$.

We use carets in this way to denote things obtained by this process of linear extension.

Remark. If $\hat{\sigma}$ is degenerate and t is of covariant type, then $\hat{t}$ may be undefined. We now arrange things, however, so that this will not matter.

Let us introduce our notion of convergence for sequences of local PL tensors.

Definition 3.4. A *sequence converging to* $p \in M$ is a sequence $(\sigma_i, t_i, \varphi_i)$ such that $\operatorname{Im}(\varphi_i) \to p$ and for each chart containing p, for all but finitely many i, the fatnesses satisfy $\theta(\hat{\sigma}_i) \geqslant b > 0$ for some b which may depend on the chart but is independent of i. Here the fatness is measured with respect to the coordinate metric.

For any particular local PL tensor, there are many charts in which $\hat{\sigma}$ is degenerate. Thus our formulation is somewhat counterintuitive unless one keeps the idea of sequences firmly in mind at all times. Observe that the t_i play no role in this definition; they have been included from the beginning for notational convenience. We discuss their convergence later on (Definition 3.9 and following).

The following result shows that the existence of a fatness bound does not depend on the choice of local coordinates.

LEMMA 3.5. *If the fatness condition of Definition 3.4 is satisfied in one chart containing p, then it is satisfied in any chart containing p.*

Proof. Let $\sigma_i \to 0 \in \mathbf{R}^D$, $\theta(\sigma_i) \geqslant b > 0$, and χ be a C^1 change of coordinates in $\mathbf{R}^D$. Slightly abusing the notation of Definition 3.3, denote by $\hat{\chi}(\sigma_i)$ the simplices formed by mapping the vertices of each σ_i with χ and extending linearly to all of each σ_i. It suffices to show that all but finitely many $\hat{\chi}(\sigma_i)$ have a fatness bound as desired.

We require

LEMMA 3.6. *Let U be an open set containing* 0 *in* $\mathbf{R}^D$ *and let* χ *be a* C^1 *diffeomorphism* $\chi: U \to V \subset \mathbf{R}^D$ *such that* $\chi(0) = 0$. *For any simplex* $\sigma = (v_0, ..., v_D)$ *in U, denote by* $\hat{\chi}$ *the affine map extending* $\chi | \{v_0, ..., v_D\}$ *and let* L^σ *be its linear part, so that* $L^\sigma(a - b) = \hat{\chi}(a) - \hat{\chi}(b)$. *For every* $\varepsilon > 0$ *there exists a neighbourhood of* 0, $N_\varepsilon \subseteq U$, *such that for every simplex* $\sigma \subseteq N_\varepsilon$ *and every edge vector* $e = v_i - v_j$ *of* σ, $|(L^\sigma - \chi_{*0})(e)|_\infty < \varepsilon \, |e|_\infty$.

Proof. Recalling that all norms are equivalent on finite-dimensional vector spaces, observe that $L^\sigma(e) = \chi(v_i) - \chi(v_j)$ and apply the approximation lemma [2, p. 377] to χ. ∎

To complete the proof of Lemma 3.5, apply Lemma 2.7 with γ the coordinate metric on $\mathbf{R}^D$, $g_{(i)}$ the constant sequence $(\chi^* \gamma)_0$, and $A_{(i)}$ the affine map $\hat{\chi}$ on σ_i. The verification that the inequality in the hypothesis of Lemma 2.7 is satisfied here is a straightforward application of the triangle and Cauchy-Schwarz inequalities, which we leave to the reader. ∎

Remark. Examining the proof of Lemma 3.5, we see that the force of the fatness bounds is that it allows us to convert the estimate of Lemma 3.6 into convergence $L^{\sigma_i} \to \chi_{*0}$ for appropriate sequences.

Now we introduce the fundamental global (meaning, on all of M) objects.

DEFINITION 3.7. A global PL tensor on M is a collection $\mathscr{T} = \{(\sigma, t, \varphi)\}$ in which there is a sequence converging to any point of M and all t are of the same type.

Note that this includes sequences of PF approximations as in Definition 2.1, but that the simplicial complexes are inessential.

Next, we investigate the behavior of global PL tensors under a change of local coordinates.

PROPOSITION 3.8 *Let $\mathscr{T}$ be a global PL tensor on a smooth manifold M, p a point in M, $(\sigma_i, t_i, \varphi_i)$ a sequence converging to p, and $\psi\colon U \to \mathbf{R}^D$ a chart at p. Let $\psi'\colon U' \to \mathbf{R}^D$ be another chart at p with change of coordinate function $\chi\colon \psi(U \cap U') \to \psi'(U \cap U')$. Denote by $\hat{t}_i$* and *$\hat{t}'_i$, respectively, the tensors on $\mathbf{R}^D$ induced as in Definition* 3.3. *If $\hat{t}_i \to t_\psi$, a constant tensor on $\mathbf{R}^D$, then $\hat{t}'_i \to t_{\psi'}$ and $\chi_{*\psi(p)} t_\psi = t_{\psi'}$.*

Proof. We suppress $\psi(p)$. Since $\hat{t}_i \to t_\psi$, $\chi_* \hat{t}_i = \chi_*(\psi \circ \varphi_i)^\wedge_* t_i \to \chi_* t_\psi$. Now, $\hat{t}'_i = (\psi' \circ \varphi_i)^\wedge_* t_i = (\chi \circ \psi \circ \varphi_i)^\wedge_* t_i = \hat{\chi}_*(\psi \circ \varphi_i)^\wedge_* t_i$. Comparing χ_* and $\hat{\chi}_*$ via Lemma 3.6, the conclusions follow. ∎

We now give the important Cauchy criterion for our fundamental global objects. We then verify that it yields good convergence behavior, as one would hope.

DEFINITION 3.9. A global PL tensor $\mathscr{T}$ is said to be *Cauchy* if and only if for every $p \in M$, every chart at p, every sequence in $\mathscr{T}$ converging to p, and every $\varepsilon > 0$ there exists N such that $i, j > N$ implies $|\hat{t}_i - \hat{t}_j|_\infty < \varepsilon$.

Here we use the notation of Definition 3.3.

PROPOSITION 3.10. *Let $\mathscr{T}$ be a Cauchy global PL tensor on a smooth manifold M, p a point in M, $\psi\colon U \to \mathbf{R}^D$ a chart at p, and $(\sigma_i, t_i, \varphi_i)$ a sequence converging to p. Denote the tensors on $\mathbf{R}^D$ induced as in Definition* 3.3 *by $\hat{t}_i$. Then $\hat{t}_i \to t$ for some constant tensor t on $\mathbf{R}^D$. Let $(\sigma'_i, t'_i, \varphi'_i)$ be another sequence converging to p with $\hat{t}'_i \to t'$ in obvious notation. Then $t = t'$.*

Proof. Construct a third sequence converging to p whose members alternate the $(\sigma_i, t_i, \varphi_i)$ with the $(\sigma'_i, t'_i, \varphi'_i)$. Then apply Definition 3.9. ∎

Proposition 3.8 gives the correct change of variables formula for the coordinate representation of a tensor. Therefore

COROLLARY 3.11. *A Cauchy global PL tensor $\mathscr{T}$ on M defines a unique, possibly discontinuous, tensor t on M.*

Any Cauchy criterion which yields convergence should have a companion uniform Cauchy criterion which yields well-behaved convergence. Here is ours, followed by the verification of good behavior.

DEFINITION 3.12. A global PL tensor $\mathscr{T}$ is uniformly Cauchy if and only if for every $p \in M$, every chart U at p, and every $\varepsilon > 0$ there exists a neighborhood of p, $V \subseteq U$, such that for all $\mathrm{Im}(\varphi_1)$, $\mathrm{Im}(\varphi_2) \subseteq V$, $|\hat{t}_i - \hat{t}_2|_\infty < \varepsilon$.

PROPOSITION 3.13. *Uniformly Cauchy implies Cauchy.*

THEOREM 3.14. *If t is the limit of a uniformly Cauchy $\mathscr{T}$, then t is continuous.*

Proof. Let $p \in M$ and $\varepsilon > 0$ and choose V corresponding to $\varepsilon/3$ as in Definition 3.12. For $q \in V$, regard $t(p)$ and $t(q)$ as constant tensors on $\mathbf{R}^D$, and use Proposition 3.10 and Definition 3.12 to conduct an $\varepsilon/3$ argument. ∎

Finally, we show that the particular sequences of metric triangulations obtained from [3] are uniformly Cauchy. In fact, our result is much stronger than this, as it applies to many other cases as well.

As in Theorem 2.2, let $f(\eta)$ be any function which is $o(\eta)$.

THEOREM 3.15. *If $\mathscr{G}$ is the set of all local PF approximations to (M, g) with error $f(\eta_\sigma)$ and such that $\theta(g_\sigma) \geqslant B > 0$*, then $\mathscr{G}$ is uniformly Cauchy and converges to g.

Proof. The estimates in the proof of Lemma 2.7 generalise to apply to all local PF approximations contained within some neighbourhood of p, instead of just a sequence. The bound required in the hypothesis of Lemma 2.7 is supplied by Lemma 2.4, which is itself a uniform statement. The neighbourhood required in Definition 3.12 is the one obtained from Lemma 2.4 for a suitable choice of ε there. ∎

4. EXAMPLES

In the first five examples we study sequences which do not have a fatness bound. We hope to convince the reader that this bound is an essential part of our formulation by exhibiting a variety of pathologies which occur in its absence. The last example gives one way of constructing a global PL tensor which converges to an arbitrary pseudo-Riemannian metric tensor.

EXAMPLE 4.1. Let σ_i be the simplices in $\mathbf{R}^2$ with vertices $(0, 0)$ and $(\pm 1/i, 1/i^2)$ for $i \geq 1$. Observe that this sequence converges to $(0, 0)$ and has no fatness bound. Consider the change of coordinates given by $x \mapsto x$ and $y \mapsto y - x^2$. If χ denotes the change of coordinates map, then all the simplices $\hat{\chi}(\sigma_i)$ are degenerate in the new coordinates. Observe that in the notation of Lemma 3.6, $\chi_{*0} = I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ but $L^{\sigma_i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, projection on the x-axis.

EXAMPLE 4.2. Continue with the same simplices as before, but now consider the change of coordinates χ given by $x \mapsto x$ and $y \mapsto y - 2x^2$. Now the images are all nondegenerate simplices, but another problem arises. Let t denote any fixed, constant vector field on $\mathbf{R}^2$ and restrict it to each σ_i, obtaining t_i. Again we have $\chi_{*0} = I_2$, but now $L^{\sigma_i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, reflection across the x-axis. The $\hat{t}_i = L^{\sigma_i}(t_i)$ converge, but not to $\chi_{*0} t$. Cf. Proposition 3.8.

EXAMPLE 4.3. We use the same σ_i, but the relevant sequence for this example is $\sigma_i^\dagger = Q^i \sigma_i$, where $Q = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, a counterclockwise rotation through 90°. We continue with t as before, define $t_i^\dagger$ by restriction, and use the same change of coordinates χ. Now we find $L^{Q^i \sigma_i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ for $i \equiv 0 (\text{mod } 2)$
and

$$L^{Q^i \sigma_i} = \begin{pmatrix} 1 & 0 \\ 2/i^2 & 1 \end{pmatrix} \to I_2, \qquad i \equiv 1 \ (\text{mod } 4),$$

$$L^{Q^i \sigma_i} = \begin{pmatrix} 1 & 0 \\ -2/i^2 & 1 \end{pmatrix} \to I_2, \qquad i \equiv 3 \ (\text{mod } 4).$$

Then the $\hat{t}_i^\dagger$ do not converge at all, due to the differing behavior for even and odd i. Cf. Proposition 3.8 again.

This next example studies the adequacy of our definition of PF approximation. The original construction is due to H. A. Schwarz; see [9], for example.

EXAMPLE 4.4. We consider a cylinder of radius 1 and height 1 as the quotient of a plane rectangle of length 2π and height 1. We divide the rectangle into mn subrectangles, dividing into m subdivisions vertically and n horizontally. Triangulate by drawing the diagonals in each subrectangle and marking the corners and centers as vertices. Let $u = (2\pi, 0)$ and $v = (0, 1)$ be tangent vectors. Following our notation in Theorem 2.2, we denote by $g_{(mn)}$ the PF metric on approximation (mn).

For the upper and lower triangles, it is straightforward to calculate

$$g_{(mn)}(u, u) = 4n^2 \sin^2(\pi/n) \to 4\pi^2,$$

$g_{(mn)}(u, v) = 0$, and

$$g_{(mn)}(v, v) = 1 - 4m^2 \sin^2(\pi/n) + 16m \sin^2(\pi/2m).$$

The problem then is to evaluate the limit for $g_{(mn)}(v, v)$. A somewhat more tedious calculation yields

$$g_{(mn)}(v, v) \to \begin{cases} 1, & m < n^2, \\ 1 + k\pi^4, & m^2 = kn^4, \\ +\infty, & m > n^2. \end{cases}$$

For the right and left triangles, $g_{(mn)}(v, v) = 1$, $g_{(mn)}(u, v) = 0$, and

$$g_{(mn)}(u, u) = 16n^2 \sin^2(\pi/2n) \to 4\pi^2.$$

Calculation of the area of the approximation from these metric tensors now agrees with the calculation from the formula of Schwarz,

$$A_{(mn)} = 2n \sin \frac{\pi}{2n} + \left[\frac{1}{4} + \frac{4m^2}{n^4}\left(n \sin \frac{\pi}{2n}\right)^4\right]^{1/2} 2n \sin \frac{\pi}{n}.$$

Taking the limit, we find

$$A_{(mn)} \to \begin{cases} 2\pi, & m < n^2, \\ \pi(1 + \sqrt{1 + k\pi^4}), & m^2 = kn^4, \\ +\infty, & m > n^2. \end{cases}$$

Thus we see that the metric tensors converge precisely when the volume formula of Schwarz converges and to the "correct" limit. Schwarz's example shows that the area may not converge to the Euclidian value 2π. From our perspective, this is due to the faulty convergence of the $g_{(mn)}$. The metrics are obtained from the secant approximation, hence satisfy $|\sigma^1| - d(v_0, v_1) = O(|\sigma^1|^3)$, so they obey all the requirements of PF approximation *except* fatness bounds.

The preceding examples were PF approximations of flat space. We now show that the pathologies exhibited were not due to the term $o(\eta)$ in Definitions 2.1 or 3.2.

EXAMPLE 4.5. Consider the 3-dimensional non-Euclidean hyperbolic space, $\mathbf{R}^3$ with the Riemannian metric of constant curvature -1. Let p_1, p_2, p_3 be any three points in general position in the hyperbolic plane $\mathbf{R}^2 \subset \mathbf{R}^3$ such that the origin p_0 lies in the interior of their convex hull. Now let $\sigma = \{v_0, v_1, v_2, v_3\}$ be a simplex and define the map φ by $v_i \mapsto p_i$. One can check that there is no flat positive-definite metric on σ which assigns to each edge (v_i, v_j) the hyperbolic distance $d(p_i, p_j)$. The simplex does actually embed in Minkowski 3-space, so in a sense, this local PF approximation has the wrong signature compared to the manifold which it is supposed to be approximating. We can certainly consider sequences of such local PF approximations, where the vertices converge to the origin. Since all members of the sequence have the wrong signature, they cannot converge to the target metric in any coordinates whatsoever. Using the coordinates from $\mathbf{R}^3$, it is clear that the coordinate simplex $\hat{\sigma}$ is degenerate, so any such sequence certainly does not have a fatness bound. The bizarre behavior of such a sequence is due to the lack of a fatness bound.

EXAMPLE 4.6. By using a moving bump-function, it is easy to produce examples which are Cauchy but not uniformly Cauchy.

Finally, we give an extension of Theorem 3.15 to nondegenerate indefinite metric tensors of arbitrary signature, or briefly, pseudo-Riemannian metrics.

EXAMPLE 4.7. Let M be a manifold with a pseudo-Riemannian metric g. Choose an arbitrary auxiliary Riemannian metric h and consider the set $\mathscr{H}$ of all local PF approximations to (M, h) with a uniform fatness bound and error function as in Theorem 3.15. For each $(\sigma, h_\sigma, \varphi) \in \mathscr{H}$ in which φ maps all the vertices into a convex normal neighborhood for the Levi-Cività connection of g, we define a constant pseudo-Riemannian metric g_σ on σ using an analogous definition of local PF approximation. Namely, for each edge (v_i, v_j) of σ, set

$$g_\sigma(v_i - v_j) = \int_0^1 g_{kl} \xi^k \xi^l \, dt,$$

the energy, integrating with respect to an affine parameter along the geodesic segment from $\varphi(v_i)$ to $\varphi(v_j)$. Here we have used the notation of Lemma 2.4. Observe that g_σ so defined is unique. Now define $\mathscr{G}$ to be the set of all such $(\sigma, g_\sigma, \varphi)$. Lemma 2.4 remains valid for a pseudo-Riemannian metric h_V as long as one replaces d^2 everywhere by the value of the appropriate energy integral.

Indeed, examining the proof of Lemma 2.4, one sees that it suffices to obtain estimate (*) there. To do this, use the Theorem of Whitehead [7, p. 73] and choose the V'' of Lemma 2.4 to be a normal neighbourhood of each of its points with respect to the Levi-Cività connection of h_V. Considered as a function of two variables, the exponential map exp of this linear connection is a diffeomorphism

$$\Phi: \bigcup_{p \in V''} \exp_p^{-1}(V'') \to V'' \times V''.$$

Now apply the approximation lemma from [2] to Φ, using $x - \xi = \exp_{\chi(a)}(\xi) - \exp_{\chi(a)}(0) - \exp_{\chi(p)*}(\xi)$. Using the theorem of Whitehead again, we obtain the desired $V' \subset V''$.

It then follows that the global PL tensor $\mathscr{G}$ is uniformly Cauchy and converges to g.

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